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The Seventeen Wallpaper Groups

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Abstract

Wallpaper patterns are categorised into lattices. A case-by-case analysis of each group of orthogonal transformations which preserve a lattice is used to develop the seventeen wallpaper groups. All the elements of each wallpaper group are described and their uniqueness is demonstrated.

1 Introduction

A *wallpaper pattern* is a two dimensional repeating pattern that fills out the whole plane [1]. The symmetry of such patterns is measured by what we will define in the next section as wallpaper groups. The fact that there are 17 such wallpaper groups was known to 19th-century crystallographers but various mathematical derivations of this fact have been presented relatively recently. This paper seeks to give added rigour and explanation to the derivation given in [1]. In particular, we leave no ambiguity as to how isomorphic wallpaper groups are related, and describe all the elements of each unique wallpaper group. Our approach is more geometric in nature than [4] yet it none-the-less predominantly algebraic. We first determine the possible basic parallelogram units from which a wallpaper pattern may be constructed. Then by considering all combinations of orthogonal transformations which preserve such a unit, we construct the corresponding wallpaper groups. We show there are precisely 17 unique such groups.

2 Preliminaries

Definition 2.1 A function $g : \mathbb{R}^2 \to \mathbb{R}^2$ belongs to the *Euclidean group* E_2 if it preserves distance, and is called an *isometry* of the plane.

Definition 2.2 *Translation* by the vector $\mathbf{v}$ is the function $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\tau(\mathbf{x}) = \mathbf{v} + \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$.

Theorem 2.3 The translations make up a subgroup T of E_2.

Proof. Refer to pp 137-138 of [1]. $\square$

Definition 2.4 Let O_2 denote the subgroup of E_2 which consists of orthogonal transformations in $\mathbb{R}^2$, i.e. 2×2 orthogonal matrices that represent rotations about the origin and reflections in lines through the origin.

Theorem 2.5 An isometry can be written in *only one* way as an orthogonal transformation followed by a translation.

Proof. Refer to pp 138 of [1]. $\square$

Definition 2.6 We denote an isometry g by an ordered pair, $g = (\mathbf{v}, M)$. This means $g(\mathbf{x}) = \mathbf{v} + M\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$, where $\mathbf{v} \in \mathbb{R}^2$ and $M \in O_2$. Let $(\mathbf{v}_1, M_1), (\mathbf{v}_2, M_2) \in E_2$, then multiplication is defined by:

$$(\mathbf{v}_1, M_1)(\mathbf{v}_2, M_2) = (\mathbf{v}_1 + M_1\mathbf{v}_2, M_1M_2)$$

Let $A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, B_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$. Then we define the simplest isometries as follows.
(a) A *translation* by vector $\mathbf{v}$ is $(\mathbf{v}, I)$ where $I = A_0$.
(b) An *anticlockwise rotation* by θ about its centre, $\mathbf{c}$, is $(\mathbf{c} - A_\theta\mathbf{c}, A_\theta)$. Note that if $A_\theta = -I$ we call the rotation a *half-turn*.
(c) A *reflection* through a line m with slope $\frac{\phi}{2}$ is $(2\mathbf{a}, B_\phi)$ where $B_\phi\mathbf{a} = -\mathbf{a}$.
(d) A *glide* composed of a reflection in the line m and a translation parallel to that line is $(2\mathbf{a} + \mathbf{b}, B_\phi)$ where $B_\phi\mathbf{a} = -\mathbf{a}$ and $B_\phi\mathbf{b} = \mathbf{b} \neq 0$.

Definition 2.7 Let G be a subgroup of E_2. Define $\pi : E_2 \to O_2$ by $\pi(\mathbf{v}, M) = M$.

Definition 2.8 The *translation subgroup* H of G is $G \cap T$. The *point group* J of G is $\pi(G)$, the group of all orthogonal matrices corresponding to the isometries in G.

Definition 2.9 A subgroup of E_2 is a *wallpaper group* G if its translation subgroup H is generated by two independent vectors and its point group J is finite.

Now we can define a glide to be *nontrivial* if it cannot be composed of a translation and a reflection in G. Otherwise it is *trivial*.

Theorem 2.10 An isomorphism between wallpaper groups takes translations to translations, rotations to rotations, reflections to reflections and glides to glides.

Proof. Refer to Theorem 25.5 of [1]. $\square$

Corollary 2.11 If two wallpaper groups are isomorphic then their point groups are also isomorphic.

Proof. Refer to Corollary 25.6 of [1]. $\square$

Definition 2.12 Let L be the orbit of the origin under the action of H on $\mathbb{R}^2$. That is, $L = \{\mathbf{v} : (\mathbf{v}, I) \in H\}$.

Since H is generated by two independent vectors, L must contain two independent vectors. Select a non-zero vector $\mathbf{a}$ of minimum length in L and, of all the vectors skew to $\mathbf{a}$, choose $\mathbf{b}$ to have the minimal length.

Theorem 2.13 The set L is the *lattice* spanned by $\mathbf{a}$ and $\mathbf{b}$. That is, L consists of all linear combinations $m\mathbf{a} + n\mathbf{b}$ where $m, n \in \mathbb{Z}$.

Proof. We first show that $m\mathbf{a} + n\mathbf{b} \in L$; $m, n \in \mathbb{Z}$. The correspondence $\psi : T \to \mathbb{R}^2$ defined by $\psi((\mathbf{v}, I)) = \mathbf{v}$, where $\mathbf{v} \in \mathbb{R}^2$, is an isomorphism: Let $(\mathbf{v}_1, I)$, $(\mathbf{v}_2, I) \in T$. Then

$$\psi((\mathbf{v}_1, I)(\mathbf{v}_2, I)) = \psi((\mathbf{v}_1 + \mathbf{v}_2, I)) = \mathbf{v}_1 + \mathbf{v}_2 = \psi((\mathbf{v}_1, I)) + \psi((\mathbf{v}_2, I))$$

so ψ is a homomorphism. ψ is injective since

$$\psi((\mathbf{v}_1, I)) = \psi((\mathbf{v}_2, I)) \Rightarrow \mathbf{v}_1 = \mathbf{v}_2$$

If $\psi((\mathbf{v}, I)) = \mathbf{x}$ where $(\mathbf{v}, I) \in T$ and $\mathbf{x} \in \mathbb{R}^2$ then $(\mathbf{v}, I) = (\mathbf{x}, I) \in T$ so ψ is onto. Now, clearly ψ sends H to L, so because H is a subgroup of T, L is a subgroup of $\mathbb{R}^2$. Hence, since L must be closed, $\mathbf{a}, \mathbf{b} \in L \Rightarrow m\mathbf{a} + n\mathbf{b} \in L$ where $m, n \in \mathbb{Z}$. The points on this lattice form a pattern of parallelograms in $\mathbb{R}^2$, as illustrated in Figure 1 of the Appendix. Finally, we show L consists only of those points on the lattice, $m\mathbf{a} + n\mathbf{b}$. Suppose $\mathbf{x} \in L$ yet is not on the lattice (i.e., is not spanned by $\mathbf{a}$ and $\mathbf{b}$). Let $\mathbf{c}$ be the closest corner of the parallelogram containing $\mathbf{x}$ to $\mathbf{x}$ and let $\mathbf{x}_m$ be the midpoint of that parallelogram. Then $\mathbf{c} = k_1\mathbf{a} + k_2\mathbf{b}$ for some $k_1, k_2 \in \mathbb{Z}$ and $\mathbf{x} - \mathbf{c}$ is not equal to $\mathbf{0}$, $\mathbf{a}$ nor $\mathbf{b}$. Without loss of generality, assume $\mathbf{b}$ is in the first quadrant, and let $\mathbf{a} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} y \\ z \end{pmatrix}$ where $x, y, z > 0$. The maximum possible value of $||\mathbf{x} - \mathbf{c}||$ is the distance from $\mathbf{x}_m$ to the vertex closest to it. That is,

$$||\mathbf{x} - \mathbf{c}|| \leq \min\{||\mathbf{x}_m - \mathbf{0}||, ||\mathbf{x}_m - \mathbf{a} - \mathbf{b}||, ||\mathbf{x}_m - \mathbf{a}||, ||\mathbf{x}_m - \mathbf{b}||\}$$

But $||\mathbf{x}_m|| = ||\mathbf{x}_m - \mathbf{a} - \mathbf{b}||$, $||\mathbf{x}_m - \mathbf{a}|| = ||\mathbf{x}_m - \mathbf{b}||$, and $||\mathbf{x}_m - \mathbf{a}|| = ||\mathbf{x}_m|| - 2xy < ||\mathbf{x}_m||$, since $xy > 0$, so that $\min\{||\mathbf{x}_m - \mathbf{0}||, ||\mathbf{x}_m - \mathbf{a} - \mathbf{b}||, ||\mathbf{x}_m - \mathbf{a}||, ||\mathbf{x}_m - \mathbf{b}||\} = ||\mathbf{x}_m - \mathbf{a}||$. Hence $||\mathbf{x} - \mathbf{c}|| \leq ||\mathbf{x}_m - \mathbf{a}||$. Now,

$$\begin{aligned} &||\mathbf{b}|| = y^2 + z^2 > ||\mathbf{a}|| = x^2 > x^2 - 2xy \Rightarrow \tfrac{1}{2}(y^2 + z^2) > \tfrac{1}{2}(x^2 - 2xy) \\ &\Rightarrow ||\mathbf{b}|| = y^2 + z^2 > \tfrac{1}{2}(x^2 - 2xy + y^2 + z^2) = ||\mathbf{x}_m - \mathbf{a}|| \geq ||\mathbf{x} - \mathbf{c}|| \end{aligned}$$

so that $||\mathbf{b}|| > ||\mathbf{x} - \mathbf{c}||$. Thus, by our choice of $\mathbf{b}$, $\mathbf{x} - \mathbf{c}$ must not be skew to $\mathbf{a}$, i.e. $\mathbf{x} - \mathbf{c} = t\mathbf{a}$ for some $t \in \mathbb{R}$. Since $\mathbf{a}$ is of minimum length in L, we must have

$$||\mathbf{a}|| \leq ||\mathbf{x} - \mathbf{c}|| = t||\mathbf{a}|| \Rightarrow t \geq 1$$

But then

$$\mathbf{x} = t\mathbf{a} + \mathbf{c} = (k_1 + t)\mathbf{a} + k_2\mathbf{b},\ t \geq 1$$

so that $\mathbf{x}$ is contained in a parallelogram where $\mathbf{c}$ is not a corner. This is a contradiction, so no such $\mathbf{x}$ exists in L, and L is spanned by $\mathbf{a}$ and $\mathbf{b}$. □

By considering all possible inequalities relating $||\mathbf{a}||, ||\mathbf{b}||, ||\mathbf{a} - \mathbf{b}||$ and $||\mathbf{a} + \mathbf{b}||$, corresponding to all the possible basic parallelograms determined by the vectors $\mathbf{a}$ and $\mathbf{b}$, we arrive at five different types of lattices.

3 The Five Types of Lattices

Recall that $||\mathbf{a}|| \leq ||\mathbf{b}||$. Placing the further condition that we replace $\mathbf{b}$ by $-\mathbf{b}$ if necessary to ensure $||\mathbf{a} - \mathbf{b}|| \leq ||\mathbf{a} + \mathbf{b}||$, there are only five possible cases.

(i) Oblique lattice, L_1 $||\mathbf{a}|| < ||\mathbf{b}|| < ||\mathbf{a} - \mathbf{b}|| < ||\mathbf{a} + \mathbf{b}||$
(ii) Rectangular lattice, L_2 $||\mathbf{a}|| < ||\mathbf{b}|| < ||\mathbf{a} - \mathbf{b}|| = ||\mathbf{a} + \mathbf{b}||$
(iii) Centred Rectangular lattice, L_3
(a) $||\mathbf{a}|| < ||\mathbf{b}|| = ||\mathbf{a} - \mathbf{b}|| < ||\mathbf{a} + \mathbf{b}||$
(b) $||\mathbf{a}|| = ||\mathbf{b}|| < ||\mathbf{a} - \mathbf{b}|| < ||\mathbf{a} + \mathbf{b}||$
(iv) Square lattice, L_4 $||\mathbf{a}|| = ||\mathbf{b}|| < ||\mathbf{a} - \mathbf{b}|| = ||\mathbf{a} + \mathbf{b}||$
(v) Hexagonal lattice, L_5 $||\mathbf{a}|| = ||\mathbf{b}|| = ||\mathbf{a} - \mathbf{b}|| < ||\mathbf{a} + \mathbf{b}||$
The lattices are illustrated in Figure 2 of the Appendix. If necessary, we will identify $\mathbf{a}$ and $\mathbf{b}$ with respect to a particular lattice L_n as $\mathbf{a}_n$ and $\mathbf{b}_n$, respectively.

We will demonstrate the existence of 17 wallpaper groups using the following limitation on the point group J.

Theorem 3.1 The point group J preserves the lattice L. That is, if $M \in J$ and $\mathbf{x} \in L$, then $M\mathbf{x} \in L$.

Proof. $\pi : G \to J$ is a homomorphism since if $(\mathbf{v}_1, M_1), (\mathbf{v}_2, M_2) \in G$,

$$\pi\left((\mathbf{v}_1, M_1)(\mathbf{v}_2, M_2)\right) = M_1M_2 = \pi\left((\mathbf{v}_1, M_1)\right)\pi\left((\mathbf{v}_2, M_2)\right).$$

Also, $\pi((\mathbf{v}, I)) = I$ so that $H = \text{kernel}(\pi)$. Suppose $M \in J$, $\mathbf{x} \in L$ and let $g = (\mathbf{v}, M) \in G$, $\tau = (\mathbf{x}, I) \in H$. Then $\pi(g\tau g^{-1}) = I$ so that $g\tau g^{-1} \in H$. Thus

$$\begin{aligned} g\tau g^{-1} &= (\mathbf{v}, M)(\mathbf{x}, I)(-M^{-1}\mathbf{v}, M^{-1}) \\ &= (M\mathbf{x}, I) \end{aligned}$$

implies $(M\mathbf{x}, I) \in H$ so that $M\mathbf{x} \in L$. $\square$

4 The Seventeen Wallpaper Groups

For each lattice type, we determine which rotations and reflections preserve the lattice. These form the group for which the point group is a subgroup. From here, each wallpaper group is classified by exhausting all possible cases: for each combination of a lattice L and a particular point group we will construct the corresponding group. Theorem 2.5 implies that every element in G can be written in only one way as the product of a translation and an orthogonal transformation. Therefore any element $g \in G$, such that $\pi(g) = M$, is included in the coset $H(\mathbf{v}, M)$ where $(\mathbf{v}, M)$ represents the realisation[1] of M from J in G. Thus $G = H \cup H(\mathbf{v}_1, M_1) \cup H(\mathbf{v}_2, M_2) \cup \ldots \cup H(\mathbf{v}_n, M_n)$ where $J = \{I, M_1, \ldots, M_n\}$, $v_i \in \mathbb{R}^2$, $M_i \in J$ and $(\mathbf{v}_i, M_i)$ represents the realisation of M_i from J in G for $i = 1, \ldots, n$ [3]. At the end, we will prove that all groups derived are mutually non-isomorphic. For an illustration of the different wallpaper groups, refer to Figure 3 of the Appendix.

The following lemma will allow us to derive all the possible rotations and reflections that preserve a particular lattice, and hence derive the possible point groups for the lattice.

Lemma 4.1 The point group preserves the lattice if and only if it preserves $\mathbf{a}$ and $\mathbf{b}$. That is, if $M \in J$ and $\mathbf{x} \in L$, then $M\mathbf{x} \in L$ if and only if $M\mathbf{a} \in L$ and $M\mathbf{b} \in L$.

Proof. Let $\mathbf{x} = k_1\mathbf{a} + k_2\mathbf{b}$ for some $k_1, k_2 \in \mathbb{Z}$. If $M\mathbf{x} \in L$ then $M\mathbf{x} = M(k_1\mathbf{a} + k_2\mathbf{b}) = c_1\mathbf{a} + c_2\mathbf{b}$ for some $c_1, c_2 \in \mathbb{Z}$. Taking $(k_1, k_2) = (1, 0)$ and $(k_1, k_2) = (0, 1)$ give $M\mathbf{a} = c_1\mathbf{a} + c_2\mathbf{b}$ and $M\mathbf{b} = c_1\mathbf{a} + c_2\mathbf{b}$ so that $M\mathbf{a} \in L$ and $M\mathbf{b} \in L$ respectively. If $M\mathbf{a} \in L$ and $M\mathbf{b} \in L$ then $M\mathbf{a} = k\mathbf{a} + l\mathbf{b}$ and $M\mathbf{b} = m\mathbf{a} + n\mathbf{b}$ for some $k, l, m, n \in \mathbb{Z}$. Thus $M\mathbf{x} = M(k_1\mathbf{a} + k_2\mathbf{b}) = k_1(k\mathbf{a} + l\mathbf{b}) + k_2(m\mathbf{a} + n\mathbf{b}) = (k_1k + k_2m)\mathbf{a} + (k_1l + k_2n)\mathbf{b} \in L$. $\square$

Lemma 4.2 The wallpaper group G corresponding to the lattice L with point group J is isomorphic to the wallpaper group G_1 which corresponds to the lattice L rotated clockwise by θ with point group $J_1 = \{A_{-\theta}MA_\theta : M \in J\}$.

Proof. We show that the correspondence $\varphi : G \to G_1$ defined by

$$\varphi((\mathbf{v}, M)) = (A_{-\theta}\mathbf{v}, A_\theta MA_{-\theta})$$

is an isomorphism.

φ is a homomorphism:

Let $(\mathbf{v}_1, M_1)$ and $(\mathbf{v}_2, M_2) \in G$. Then

[1]For a formal definition, see the appendix of [3]. In this paper it is sufficient to define an element of G to *realise* $M \in J$ if it has the form $(\mathbf{v}, M)$ where $\mathbf{v} \in \mathbb{R}^2$.

$$\begin{aligned}
&\varphi((\mathbf{v}_1, M_1))\,\varphi((\mathbf{v}_2, M_2)) \\
&\quad = (A_{-\theta}\mathbf{v}_1, A_{-\theta}M_1A_\theta)(A_{-\theta}\mathbf{v}_2, A_{-\theta}M_2A_\theta) \\
&\quad = (A_{-\theta}\mathbf{v}_1 + A_{-\theta}M_1\mathbf{v}_2, A_{-\theta}M_1M_2A_\theta) \\
&\varphi((\mathbf{v}_1, M_1)(\mathbf{v}_2, M_2)) \\
&\quad = \varphi((\mathbf{v}_1 + M_1\mathbf{v}_2, M_1M_2) \\
&\quad = (A_{-\theta}\mathbf{v}_1 + A_{-\theta}M_1\mathbf{v}_2, A_{-\theta}M_1M_2A_\theta)
\end{aligned}$$

φ is injective:

$$\begin{aligned}
&\varphi((\mathbf{v}_1, M_1)) = \varphi((\mathbf{v}_2, M_2)) \\
&\quad \Rightarrow (A_{-\theta}\mathbf{v}_1, A_{-\theta}M_1A_\theta) = (A_{-\theta}\mathbf{v}_2, A_{-\theta}M_2A_\theta) \\
&\quad \Rightarrow A_{-\theta}\mathbf{v}_1 = A_{-\theta}\mathbf{v}_2 \Rightarrow \mathbf{v}_1 = \mathbf{v}_2,\ A_{-\theta}M_1A_\theta = A_{-\theta}M_2A_\theta \Rightarrow M_1 = M_2
\end{aligned}$$

φ is onto:

Let $(\mathbf{x}, N) \in G_1$ and $(\mathbf{v}, M) \in G$. Set

$$\varphi((\mathbf{v}, M)) = (\mathbf{x}, N)$$

Then $(\mathbf{v}, M) = (A_\theta\mathbf{x}, A_\theta MA_{-\theta}) \in G$.

□

Lemma 4.3 The wallpaper group G corresponding to the lattice L_n with point group J is isomorphic to the wallpaper group G_1 which corresponds to the lattice L_m, which is L_n scaled in the horizontal direction by λ and the vertical direction by μ, with point group J.

Proof. We construct vectors $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$, which are at right angles, from the vectors $\mathbf{a}_n$ and $\mathbf{b}_n$. We then proceed to scale along $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$, arriving at a new lattice spanned by $\mathbf{a}_m$ and $\mathbf{b}_m$. This is represented by the correspondence $\varphi : G \to G_1$ defined by

$$\varphi\big((\alpha\bar{\mathbf{a}} + \beta\bar{\mathbf{b}}, M)\big) = (\lambda\alpha\bar{\mathbf{a}} + \mu\beta\bar{\mathbf{b}}, M)$$

where $\bar{\mathbf{a}}, \bar{\mathbf{b}} \in \mathbb{R}^2$ with $\bar{\mathbf{a}} \perp \bar{\mathbf{b}}$; $M \in J$ and $\alpha\bar{\mathbf{a}} + \beta\bar{\mathbf{b}} = m\mathbf{a}_n + n\mathbf{b}_n$, $\lambda\alpha\bar{\mathbf{a}} + \mu\beta\bar{\mathbf{b}} = k\mathbf{a}_m + l\mathbf{b}_m$ for some $\alpha, \beta, \lambda, \mu \in \mathbb{R}$ and $k, l, m, n \in \mathbb{Z}$. We show that φ is an isomorphism.

φ is a homomorphism:

$$\begin{aligned}
&\varphi\big((\alpha_1\bar{\mathbf{a}} + \beta_1\bar{\mathbf{b}}, M_1)\big)\,\varphi\big((\alpha_2\bar{\mathbf{a}} + \beta_2\bar{\mathbf{b}}, M_2)\big) \\
&\quad = (\lambda\alpha_1\bar{\mathbf{a}} + \mu\beta_1\bar{\mathbf{b}}, M_1)(\lambda\alpha_2\bar{\mathbf{a}} + \mu\beta_2\bar{\mathbf{b}}, M_2) \\
&\quad = (\lambda(\alpha_1 + \alpha_2 s + \beta_2 v)\bar{\mathbf{a}} + \mu(\beta_1 + \alpha_2 t + \beta_2 u)\bar{\mathbf{a}}, M_1M_2) \\
&\qquad \text{where } M\bar{\mathbf{a}} = s\bar{\mathbf{a}} + t\bar{\mathbf{b}},\ M\bar{\mathbf{b}} = u\bar{\mathbf{a}} + v\bar{\mathbf{b}} \text{ for some } s, t, u, v \in \mathbb{R}. \\
&\varphi\big((\alpha_1\bar{\mathbf{a}} + \beta_1\bar{\mathbf{b}}, M_1)(\alpha_2\bar{\mathbf{a}} + \beta_2\bar{\mathbf{b}}, M_2)\big) \\
&\quad = ((\alpha_1 + \alpha_2 s + \beta_2 v)\bar{\mathbf{a}} + (\beta_1 + \alpha_2 t + \beta_2 u)\bar{\mathbf{b}}, M_1M_2) \\
&\quad = (\lambda(\alpha_1 + \alpha_2 s + \beta_2 v)\bar{\mathbf{a}} + \mu(\beta_1 + \alpha_2 t + \beta_2 u)\bar{\mathbf{b}}, M_1M_2)
\end{aligned}$$

φ is injective:

$$\begin{aligned}
&\varphi\big((\alpha_1\bar{\mathbf{a}} + \beta_1\bar{\mathbf{b}}, M_1)\big) = \varphi\big((\alpha_2\bar{\mathbf{a}} + \beta_2\bar{\mathbf{b}}, M_2)\big) \\
&\quad \Rightarrow (\lambda\alpha_1\bar{\mathbf{a}} + \mu\beta_1\bar{\mathbf{b}}, M_1) = (\lambda\alpha_2\bar{\mathbf{a}} + \mu\beta_2\bar{\mathbf{b}}, M_2) \\
&\quad \Rightarrow \alpha_1 = \alpha_2, \beta_1 = \beta_2, M_1 = M_2
\end{aligned}$$

φ is onto:

Let $(\alpha_2\bar{\mathbf{a}} + \beta_2\bar{\mathbf{b}}, N) \in G_1$ so that $\alpha_2\bar{\mathbf{a}} + \beta_2\bar{\mathbf{b}} = \lambda\alpha\mathbf{a} + \mu\beta\mathbf{b}$ for some α, $\beta \in \mathbb{R}$ and $N \in J$. Set

$$\varphi\left((\alpha_1\bar{\mathbf{a}} + \beta_1\bar{\mathbf{b}}, M)\right) = (\alpha_2\bar{\mathbf{a}} + \beta_2\bar{\mathbf{b}}, N)$$

Then $(\alpha_1\bar{\mathbf{a}} + \beta_1\bar{\mathbf{b}}, M) = (\alpha\bar{\mathbf{a}} + \beta\bar{\mathbf{b}}, N) \in G$.

Thus φ is an isomorphism. □

Theorem 4.4 Let the wallpaper group corresponding to the lattice L_n with point group J be G and let the wallpaper group corresponding to the lattice L_m with point group J_1 be G_1. Then if $M \in J$ and $A_{-\theta}MA_\theta \in J_1$ the composition of correspondences

$$\begin{aligned}&(m\mathbf{a}_n + n\mathbf{b}_n, M) \to (A_{-\theta}(m\mathbf{a}_n + n\mathbf{b}_n), A_{-\theta}MA_\theta)\\ &\quad = (\alpha\bar{\mathbf{a}} + \beta\bar{\mathbf{b}}, B_0) \to (\lambda\alpha\bar{\mathbf{a}} + \mu\beta\bar{\mathbf{b}}, A_{-\theta}MA_\theta)\\ &\qquad = (k\mathbf{a}_m + l\mathbf{b}_m, A_{-\theta}MA_\theta)\end{aligned}$$

is an isomorphism between G and G_1 where $\alpha, \beta, \lambda, \mu \in \mathbb{R}$; $k, l, m, n \in \mathbb{Z}$.

Proof. Since the composition of isomorphisms is an isomorphism, this follows directly from Lemmas 4.2 and 4.3. □

It follows from Theorem 4.4 that if we can find appropriate values for θ, $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, α, β, λ, μ, m, n, we can show two wallpaper groups are isomorphic.

Without loss of generality assume $\mathbf{a}_n = \begin{pmatrix} x \\ 0 \end{pmatrix}$, $x \in \mathbb{R}$ and $\mathbf{b}_n$ is in the first quadrant. Take $k, l, m, n \in \mathbb{Z}$.

4.1 Case (i) - The lattice of G is Oblique

We have $\mathbf{a}_1 = \begin{pmatrix} x \\ 0 \end{pmatrix}$ $\mathbf{b}_1 = \begin{pmatrix} s \\ t \end{pmatrix}$ where $0 < s < \frac{1}{2}x$; $t > \frac{\sqrt{3}}{2}x > 0$.

Theorem 4.5 If G is a wallpaper group corresponding to a pattern with an oblique lattice, then $J \subseteq \{I, -I\}$.

Proof. Possible rotations in J: I, $-I$. Suppose $A_\theta \in J$. Then

$$A_\theta\mathbf{a}_1 = \left(\cos\theta - \frac{s}{t}\sin\theta\right)\mathbf{a}_1 + \left(\frac{s}{t}\sin\theta\right)\mathbf{b}_1 = k\mathbf{a}_1 + l\mathbf{b}_1$$

And

$$A_\theta\mathbf{b}_1 = \left(-\frac{t}{s}\sin\theta - \frac{s^2}{xt}\sin\theta\right)\mathbf{a}_1 + \left(\frac{s}{t}\sin\theta + \cos\theta\right)\mathbf{b}_1 = m\mathbf{a}_1 + n\mathbf{b}_1$$

Thus we have equations

$$\cos\theta - \frac{s}{t}\sin\theta = k \ (1) \qquad \frac{x}{t}\sin\theta = l \ (2)$$
$$-\frac{t}{x}\sin\theta - \frac{s^2}{xt}\sin\theta = m \ (3) \qquad \frac{s}{t}\sin\theta + \cos\theta = n \ (4)$$

Since $\frac{x}{t} < \frac{2}{\sqrt{3}}$ and $-1 \leq \sin\theta \leq 1$, (2) implies $-1 < -\frac{2}{\sqrt{3}} < l < \frac{2}{\sqrt{3}} < 1$, so that $l = 0$. Hence $\sin\theta = 0$ since $\frac{x}{t} > 0$, so that $\theta = 0, \pi$, which satisfy all four equations.

Possible reflections in J: none. Suppose $B_\phi \in J$. Then

$$B_\phi \mathbf{a}_1 = \left(\cos\phi - \frac{s}{t}\sin\phi\right)\mathbf{a}_1 + \left(\frac{x}{t}\sin\phi\right)\mathbf{b}_1 = k\mathbf{a}_1 + l\mathbf{b}_1$$

And

$$B_\phi \mathbf{b}_1 = \left(\frac{2s}{x}\cos\phi + \frac{t}{x}\sin\phi - \frac{s^2}{xt}\sin\phi\right)\mathbf{a}_1 + \left(\frac{s}{t}\sin\phi - \cos\phi\right)\mathbf{b}_1 = m\mathbf{a}_1 + n\mathbf{b}_1$$

Thus we have equations

$$\cos\phi - \frac{s}{t}\sin\phi = k \ (5) \qquad \frac{x}{t}\sin\phi = l \ (6)$$
$$\frac{2s}{x}\cos\phi + \frac{t}{x}\sin\phi - \frac{s^2}{xt}\sin\phi = m \ (7) \qquad \frac{s}{t}\sin\phi - \cos\phi = n \ (8)$$

In a similar consideration to that for rotations, we find that $\sin\phi = 0$, so that (7) becomes $\frac{2s}{x}\cos\phi = m$. But $0 < \frac{2s}{x} < 1$ so that $-1 < m < 1$. Hence $m = 0 \Rightarrow \cos\phi = 0$. Thus no such ϕ satisfies all four equations, and we have no reflections in J.

So we have $J \subseteq \{I, -I\}$. □

Thus J is $\{I\}$ or $\{I, -I\}$ [2].

4.1.1 $J = \{I\}$

Certainly $(\mathbf{0}, I)$ is an acceptable realisation of I from J in G and we have the wallpaper group **p1**.

Elements of **p1**:

- $(m\mathbf{a}_1 + n\mathbf{b}_1, I)(\mathbf{0}, I) = (m\mathbf{a}_1 + n\mathbf{b}_1, I)$
  Translations

We show that any wallpaper group G with lattice L_n and point group $\{I\}$ is isomorphic to **p1**. The isomorphism is $\varphi : G \to \mathbf{p1}$ defined by

$$\varphi\left((m\mathbf{a}_n + n\mathbf{a}_n, I)\right) = (m\mathbf{a}_1 + n\mathbf{b}_1, I)$$

φ is a homomorphism:

$$\begin{aligned}
&\varphi((m_1\mathbf{a}_n + n_1\mathbf{a}_n, I))\,\varphi((m_2\mathbf{a}_n + n_2\mathbf{a}_n, I)) \\
&\quad = (m_1\mathbf{a}_1 + n_1\mathbf{b}_1, I)(m_2\mathbf{a}_1 + n_2\mathbf{b}_1, I) \\
&\quad = ((m_1 + m_2)\mathbf{a}_1 + (n_1 + n_2)\mathbf{b}_1, I) \\
&\varphi((m_1\mathbf{a}_n + n_1\mathbf{a}_n, I)(m_2\mathbf{a}_n + n_2\mathbf{a}_n, I)) \\
&\quad = \varphi(((m_1 + m_2)\mathbf{a}_n + (n_1 + n_2)\mathbf{b}_n, I)) \\
&\quad = ((m_1 + m_2)\mathbf{a}_1 + (n_1 + n_2)\mathbf{b}_1, I)
\end{aligned}$$

φ is injective:

$$\begin{aligned}
&\varphi((m_1\mathbf{a}_n + n_1\mathbf{b}_n, I)) = \varphi((m_2\mathbf{a}_n + n_2\mathbf{b}_n, I)) \\
&\quad \Rightarrow (m_1\mathbf{a}_1 + n_1\mathbf{b}_1, I) = (m_2\mathbf{a}_1 + n_2\mathbf{b}_1, I) \\
&\quad \Rightarrow m_1 = m_2,\ n_1 = n_2
\end{aligned}$$

φ is onto:
Let $(k\mathbf{a}_1 + l\mathbf{b}_1, I) \in \mathbf{p1}$ and $(m\mathbf{a}_n + n\mathbf{a}_n, I) \in G$. Set

$$\varphi((m\mathbf{a}_n + n\mathbf{a}_n, I)) = (k\mathbf{a}_1 + l\mathbf{b}_1, I)$$

Then $(m\mathbf{a}_n + n\mathbf{a}_n, I) = (k\mathbf{a}_n + k\mathbf{a}_n, I) \in G$.

Thus φ is an isomorphism, and this group is isomorphic to **p1**. We therefore no longer need to consider this point group as it cannot give rise to any new wallpaper groups.

4.1.2 $J = \{I, -I\}$

Choose the centre of the half-turn as origin so that $(\mathbf{0}, -I) \in G$ realises $-I$ from J in G and we have **p2**.

Elements of **p2**:

- Translations
- $(m\mathbf{a}_1 + n\mathbf{b}_1, I)(\mathbf{0}, -I) = (m\mathbf{a}_1 + n\mathbf{b}_1, -I)$
  Half-turns centred at the points $\frac{1}{2}m\mathbf{a}_1 + \frac{1}{2}n\mathbf{b}_1$

In a similar fashion to the verification of the isomorphism in Case 4.1.1, it can be shown that any wallpaper group with point group $\{I, -I\}$ is isomorphic to **p2**. Thus this point group will no longer be considered.

4.2 Case (ii) - The lattice of G is Rectangular

We have $\mathbf{a}_2 = \begin{pmatrix} x \\ 0 \end{pmatrix}$ $\mathbf{b}_2 = \begin{pmatrix} 0 \\ y \end{pmatrix}$ where $y > x > 0$.

Theorem 4.6 If G is a wallpaper group corresponding to a pattern with an oblique lattice, then $J \subseteq \{I, -I\}$.

Proof. Possible rotations in J: I, $-I$. Suppose $A_\theta \in J$. Then

$$A_\theta \mathbf{a}_2 = (\cos\theta)\,\mathbf{a}_2 + \left(\frac{x}{y}\sin\theta\right)\mathbf{b}_2 = k\mathbf{a}_2 + l\mathbf{b}_2$$

And

$$A_\theta \mathbf{b}_2 = \left(-\frac{y}{x}\sin\theta\right)\mathbf{a}_2 + (\cos\theta)\,\mathbf{b}_2 = m\mathbf{a}_2 + n\mathbf{b}_2$$

Thus we have equations

$$\begin{array}{ll} \cos\theta = k \ (9) & \frac{x}{y}\sin\theta = l \ (10) \\ -\frac{y}{x}\sin\theta = m \ (11) & \cos\theta = n \ (12) \end{array}$$

Since $0 < \frac{x}{y} < 1$, (1) implies $-1 < l < 1$, so $\theta = 0,\ \pi$; which satisfy all four equations.

Possible reflections in $J : B_0, B_\pi$. Suppose $B_\phi \in J$. Then

$$B_\phi \mathbf{a}_2 = (\cos\phi)\,\mathbf{a}_2 + \left(\frac{x}{y}\sin\phi\right)\mathbf{b}_2 = k\mathbf{a}_2 + l\mathbf{b}_2$$

And

$$B_\phi \mathbf{b}_2 = \left(\frac{y}{x}\sin\phi\right)\mathbf{a}_2 + (-\cos\phi)\,\mathbf{b}_2 = m\mathbf{a}_2 + n\mathbf{b}_2$$

Thus we have equations

$$\begin{array}{ll} \cos\phi = k \ (5) & \frac{x}{y}\sin\phi = l \ (6) \\ \frac{y}{x}\sin\phi = m \ (7) & -\cos\phi = n \ (8) \end{array}$$

Only $\phi = 0, \pi$ satisfy all four equations.

So we have $J \subseteq \{I, -I, B_0, B_\pi\}$ □

Thus J is one of: $\{I\}$, $\{I, -I\}$, $\{I, B_0\}$, $\{I, B_\pi\}$ or $\{I, -I, B_0, B_\pi\}$ [2].

4.2.1 $J = \{I, B_0\}$

Suppose $(\alpha\mathbf{a}_2 + \beta\mathbf{b}_2, B_0)$ realises B_0 from J in G and set the origin on the glide line so that $\beta = 0$. Applying the glide twice gives a translation:

$$(\alpha\mathbf{a}_2, B_0)(\alpha\mathbf{a}_2, B_0) = (2\alpha\mathbf{a}_2, I)$$

Now, a translation in G along this line has the form $(k\mathbf{a}_2, I)$ where $k \in \mathbb{Z}$ and hence $k = 2\alpha$, so $\alpha = \frac{1}{2}k$ and our glide has the form $(\frac{1}{2}k\mathbf{a}_2, B_0)$. We consider the two cases, if k is odd or even, to arrive at two different wallpaper groups.

If k is even, then $-\frac{1}{2}k \in \mathbb{Z}$ so $(-\frac{1}{2}k\mathbf{a}_2, I) \in G$. Hence

$$(-\tfrac{1}{2}k\mathbf{a}_2, I)(\tfrac{1}{2}k\mathbf{a}_2, B_0) = (\mathbf{0}, B_0) \in G$$

and we have **pm**.

Elements of **pm**:

- Translations
- $(m\mathbf{a}_2 + n\mathbf{b}_2, I)(\mathbf{0}, B_0) = \big(m\mathbf{a}_2 + 2(\frac{1}{2}n\mathbf{b}_2), B_0\big)$
  Horizontal trivial glides passing through or midway between the lattice points with translation part a multiple of $\mathbf{a}_2$

If k is odd, then $-\frac{1}{2}(k-1) \in \mathbb{Z}$ so $(-\frac{1}{2}(k-1)\mathbf{a}_2, I) \in G$ and hence

$$(-\tfrac{1}{2}(k-1)\mathbf{a}_2, I)(\tfrac{1}{2}k\mathbf{a}_2, B_0) = (\tfrac{1}{2}\mathbf{a}_2, B_0) \in G$$

and we have **pg**.

Elements of **pg**:

- Translations
- $(m\mathbf{a}_2 + n\mathbf{b}_2, I)(\frac{1}{2}\mathbf{a}_2, B_0) = \big(\frac{1}{2}(2m+1)\mathbf{a}_2 + 2(\frac{1}{2}n\mathbf{b}_2), B_0\big)$
  Horizontal nontrivial glides passing through or midway between the lattice points with translation part an odd multiple of $\frac{1}{2}\mathbf{a}_2$

4.2.2 $J = \{I, B_\pi\}$

Choosing $\theta = \frac{\pi}{2}$, $\bar{\mathbf{a}} = \mathbf{a}_2$, $\bar{\mathbf{b}} = \mathbf{b}_2$, $\alpha = n$, $\beta = -m$, $\lambda = 1$, $\mu = 1$, $k = n$, $l = -m$ in Theorem 4.4 gives an isomorphism from the resulting groups to those in Case 4.2.1.

4.2.3 $J = \{I, -I, B_0, B_\pi\}$

By the arguments for **pm** and **pg**, B_0 and B_π can be realised in G as reflections $(\mathbf{0}, B_0), (\mathbf{0}, B_\pi)$ or as nontrivial glides $(\frac{1}{2}\mathbf{a}_2, B_0), (\frac{1}{2}\mathbf{b}_2, B_\pi)$, respectively, where we have chosen the origin as the intersection of the horizontal and vertical reflections or glides. Combinations of these in G give rise to different wallpaper groups.

If $(\mathbf{0}, B_0)$ and $(\mathbf{0}, B_\pi) \in G$, then

$$(\mathbf{0}, B_0)(\mathbf{0}, B_\pi) = (\mathbf{0}, -I)$$

realises $-I$ from J in G and we have **p2mm**.

Elements of **p2mm**:

- Elements of **pm**
- $(m\mathbf{a}_2 + n\mathbf{b}_2, I)(\mathbf{0}, B_\pi) = \big(n\mathbf{b}_2 + 2(\frac{1}{2}m\mathbf{a}_2)), B_\pi\big)$
  Vertical trivial glides passing through or midway between the lattice points with translation part a multiple of $\mathbf{b}_2$
- $(m\mathbf{a}_2 + n\mathbf{b}_2, I)(\mathbf{0}, -I) = (m\mathbf{a}_2 + n\mathbf{b}_2, -I)$
  Half-turns centred at the points $\frac{1}{2}m\mathbf{a}_2 + \frac{1}{2}n\mathbf{b}_2$

If $(\mathbf{0}, B_0) \in G$ but $(\mathbf{0}, B_\pi) \notin G$, then by the argument of **pg** we have $(\frac{1}{2}\mathbf{b}_2, B_\pi) \in G$. Hence

$$(\tfrac{1}{2}\mathbf{b}_2, B_\pi)(\mathbf{0}, B_0) = (\tfrac{1}{2}\mathbf{b}_2, -I)$$

realises $-I$ from J in G and we have **p2mg**.

Elements of **p2mg**:

- Elements of **pg**
- $(m\mathbf{a}_2 + n\mathbf{b}_2, I)(\mathbf{0}, B_0) = \big(m\mathbf{a}_2 + 2(\frac{1}{2}n\mathbf{b}_2), B_0\big)$
  Horizontal trivial glides passing through or midway between the lattice points with translation part a multiple of $\mathbf{a}_2$
- $(m\mathbf{a}_2 + n\mathbf{b}_2, I)(\frac{1}{2}\mathbf{b}_2, -I) = \big(m\mathbf{a}_2 + (n + \frac{1}{2})\mathbf{b}_2, -I\big)$
  Half-turns centred at the points $\frac{1}{2}m\mathbf{a}_2 + \frac{1}{4}(2n+1)\mathbf{b}_2$

If $(\mathbf{0}, B_\pi) \in G$ but $(\mathbf{0}, B_0) \notin G$, then the isomorphism given in case 4.2.2 shows this group is isomorphic to **p2mg**.

If $(\mathbf{0}, B_0)$ and $(\mathbf{0}, B_\pi) \notin G$, then $(\frac{1}{2}\mathbf{a}_2, B_0)$ and $(\frac{1}{2}\mathbf{b}_2, B_\pi) \in G$. Hence $-I$ from J is realised in G as

$$\big(\tfrac{1}{2}\mathbf{a}_2, B_0\big)\big(\tfrac{1}{2}\mathbf{b}_2, B_\pi\big) = \big(\tfrac{1}{2}\mathbf{a}_2 - \tfrac{1}{2}\mathbf{b}_2, -I\big)$$

and we have **p2gg**.

Elements of **p2gg**:

- Elements of **pg**
- $(m\mathbf{a}_2 + n\mathbf{b}_2, I)\big(\frac{1}{2}\mathbf{b}_2, B_\pi\big) = \big(\frac{1}{2}(2n+1)\mathbf{b}_2 + 2(\frac{1}{2}m\mathbf{a}_2), B_\pi\big)$.
  Nontrivial vertical glides passing through or midway between the lattice points with translation part an odd multiple of $\frac{1}{2}\mathbf{b}_2$
- $(m\mathbf{a}_2 + n\mathbf{b}_2, I)(\frac{1}{2}\mathbf{a}_2 - \frac{1}{2}\mathbf{b}_2, -I) = \big((m + \frac{1}{2})\mathbf{a}_2 + (n - \frac{1}{2})\mathbf{b}_2, -I\big)$
  Half-turns centred at the points $\frac{1}{4}(2m+1)\mathbf{a}_2 + \frac{1}{4}(2n-1)\mathbf{b}_2$

4.3 Case (iii) - The lattice of G is Centred Rectangular

We have $\mathbf{a}_3 = \begin{pmatrix} x \\ 0 \end{pmatrix}$ $\mathbf{b}_3 = \begin{pmatrix} \frac{1}{2}x \\ t \end{pmatrix}$ where $t > \frac{\sqrt{3}}{2}x > 0$.

Theorem 4.7 If G is a wallpaper group corresponding to a pattern with a centred rectangular lattice, then $J \subseteq \{I, -I, B_0, B_\pi\}$.

Proof. Possible rotations in J: I, $-I$, B_0, B_π. Suppose $A_\theta \in J$. Then

$$A_\theta \mathbf{a}_3 = \left(\cos\theta - \frac{1}{2}\frac{x}{t}\sin\theta\right)\mathbf{a}_3 + \left(\frac{x}{t}\sin\theta\right)\mathbf{b}_3 = k\mathbf{a}_3 + l\mathbf{b}_3$$

And

$$A_\theta \mathbf{b}_3 = \left(-t\sin\theta - \frac{1}{4}\frac{x^2}{t}\sin\theta\right)\mathbf{a}_3 + \left(\frac{1}{2}\frac{x}{t}\sin\theta + \cos\theta\right)\mathbf{b}_3 = m\mathbf{a}_3 + n\mathbf{b}_3$$

Thus we have equations

$$\cos\theta - \frac{1}{2}\frac{x}{t}\sin\theta = k \ (13) \qquad \frac{x}{t}\sin\theta = l \ (14)$$
$$-t\sin\theta - \frac{1}{4}\frac{x^2}{t}\sin\theta = m \ (15) \qquad \frac{1}{2}\frac{x}{t}\sin\theta + \cos\theta = n \ (16)$$

Substituting (14) into (13) gives $\cos\theta = k + \frac{1}{2}l$ so that, since $-1 < \cos\theta < 1$, we have $\cos\theta = 0, \pm\frac{1}{2}, \pm1$, and hence $\sin\theta = \pm1, \pm\frac{\sqrt{3}}{2}, 0$. If $\sin\theta = 0$, then by (14) $l = 0$ and (13) is satisfied. If $\sin\theta = \pm\frac{\sqrt{3}}{2}$, then by (14) $\frac{x}{t} = \frac{2}{\sqrt{3}}l$. But $0 < \frac{x}{t} < \frac{2}{\sqrt{3}}$ so $0 < \frac{2}{\sqrt{3}}l < \frac{2}{\sqrt{3}}$, hence $0 < l < 1$ so that l is not an integer. Thus $\sin\theta \neq 0$. If $\cos\theta = 0$ and $\sin\theta = \pm1$, then by (13) $\frac{x}{t} = \pm2k$, so that $0 < \pm2k < \frac{2}{\sqrt{3}}$, or $0 < \pm k < \frac{1}{\sqrt{3}}$ so that k is not an integer. Thus $\sin\theta \neq \pm1$. We conclude that $\sin\theta = 0$ only so that $\theta = 0, \pi$, and these satisfy all four equations.

Possible reflections in $J : B_0, B_\pi$. Suppose $B_\phi \in J$. Then

$$B_\phi \mathbf{a}_3 = \left(\cos\phi - \frac{1}{2}\frac{x}{t}\sin\phi\right)\mathbf{a}_3 + \left(\frac{x}{t}\sin\phi\right)\mathbf{b}_3 = k\mathbf{a}_3 + l\mathbf{b}_3$$

And

$$B_\phi \mathbf{b}_3 = \left(\cos\phi + \frac{t}{x}\sin\phi - \frac{1}{4}\frac{x}{t}\sin\phi\right)\mathbf{a}_3 + \left(\frac{1}{2}\frac{x}{t}\sin\phi - \cos\phi\right)\mathbf{b}_3 = m\mathbf{a}_3 + n\mathbf{b}_3$$

Thus we have equations

$$\cos\phi - \frac{1}{2}\frac{x}{t}\sin\phi = k \ (17) \qquad \frac{x}{t}\sin\phi = l \ (18)$$
$$\cos\phi + \frac{t}{x}\sin\phi - \frac{1}{4}\frac{x}{t}\sin\phi = m \ (19) \qquad \frac{1}{2}\frac{x}{t}\sin\phi - \cos\phi = n \ (20)$$

Only $\phi = 0, \pi$ satisfy all four equations.

So we have $J \subseteq \{I, -I, B_0, B_\pi\}$ □

Thus J is one of: $\{I\}$, $\{I, -I\}$, $\{I, B_0\}$, $\{I, B_\pi\}$ or $\{I, -I, B_0, B_\pi\}$ [2].

4.3.1 $J = \{I, B_0\}$

Suppose $(\alpha\mathbf{a}_3 + \beta(2\mathbf{b}_3 - \mathbf{a}_3), B_0)$ realises B_0 in G and choose a point on the glide line as origin so that $\beta = 0$.

Then

$$(\alpha\mathbf{a}_3, B_0)(\alpha\mathbf{a}_3, B_0) = (2\alpha\mathbf{a}_3, I)$$

so that $2\alpha = k$ for some $k \in \mathbb{Z}$, or $\alpha = \frac{1}{2}k$. Thus $(\frac{1}{2}k\mathbf{a}_3, B_0) \in G$. We show that if k is even or odd we obtain the same wallpaper group.

If k is even, $-\frac{1}{2}k \in \mathbb{Z}$ so $\left(-\frac{1}{2}k\mathbf{a}_3, I\right) \in G$. Thus

$$\left(-\tfrac{1}{2}k\mathbf{a}_3, I\right)\left(\tfrac{1}{2}k\mathbf{a}_3, B_0\right) = (\mathbf{0}, B_0) \in G$$

and we have **cm**.

Elements of **cm**:

- Translations
- $(m\mathbf{a}_3 + n\mathbf{b}_3, I)(\mathbf{0}, B_0) = \left((2m+n)(\frac{1}{2}\mathbf{a}_3) + 2(\frac{1}{4}n(2\mathbf{b}_3 - \mathbf{a}_3)), B_0\right)$
  Horizontal glides which are trivial and pass through and midway between the lattice points with translation part a multiple of $\mathbf{a}_3$, or are nontrivial and pass through multiples of $\frac{1}{4}(2\mathbf{b}_3 - \mathbf{a}_3)$ with translation part an odd multiple of $\frac{1}{2}\mathbf{a}_3$

If k is odd, $-\frac{1}{2}(k+1) \in \mathbb{Z}$ so $\left(-\frac{1}{2}(k+1)\mathbf{a}_3 + \mathbf{b}_3, I\right) \in G$. Thus

$$\left(-\tfrac{1}{2}(k+1)\mathbf{a}_3 + \mathbf{b}_3, I\right)\left(\tfrac{1}{2}k\mathbf{a}_3, B_0\right) = \left(\tfrac{1}{2}(2\mathbf{b}_3 - \mathbf{a}_3), B_0\right) \in G$$

But the resulting group G is isomorphic to **cm**. The isomorphism is $\varphi : \mathbf{cm} \to G$ defined by $\varphi((\mathbf{v}, M)) = (\mathbf{v} + \frac{1}{4}(2\mathbf{b}_3 - \mathbf{a}_3) - \frac{1}{4}M(2\mathbf{b}_3 - \mathbf{a}_3), M)$ where $\mathbf{v} \in H$ and $M \in J$. Geometrically, this shifts the origin vertically by $\frac{1}{4}(2\mathbf{b}_3 - \mathbf{a}_3)$.

4.3.2 $J = \{I,\ B_\pi\}$

We develop the elements of the resulting group G for use in the next case. By a similar argument to the previous case, $(\mathbf{0}, B_\pi)$ realises B_π from J in G.

Elements of G:

- Translations
- $(m\mathbf{a}_3 + n\mathbf{b}_3, I)(\mathbf{0}, B_\pi) = \left(\frac{1}{2}n(2\mathbf{b}_3 - \mathbf{a}_3) + 2(\frac{1}{4}(2m+n))\mathbf{a}_3, B_\pi\right)$
  Vertical glides which are trivial and pass through and midway between the lattice points with translation part a multiple of $2\mathbf{b}_3 - \mathbf{a}_3$, or are nontrivial and pass through multiples of $\frac{1}{4}\mathbf{a}_3$ with translation part an odd multiple of $\frac{1}{2}(2\mathbf{b}_3 - \mathbf{a}_3)$

Choosing $\theta = \frac{\pi}{2}$, $\bar{\mathbf{a}} = 2\mathbf{b}_3 - \mathbf{a}_3$, $\bar{\mathbf{b}} = \mathbf{a}_3$, $\alpha = \frac{1}{2}n$, $\beta = \frac{1}{2}(2m+n)$, $\lambda = -\dfrac{\beta}{\alpha}$, $\mu = \dfrac{\alpha}{\beta}$, $k = -m-n$, $l = 2m+n$ in Theorem 4.4 gives an isomorphism from this group to **cm**.

4.3.3 $J = \{I,\ -I,\ B_0,\ B_\pi\}$

It follows from the previous two cases that $(\mathbf{0}, B_0)$ and $(\mathbf{0}, B_\pi) \in G$. Hence

$$(\mathbf{0}, B_0)(\mathbf{0}, B_\pi) = (\mathbf{0}, -I)$$

realises $-I$ from J in G and we have **c2mm**.

Elements of **c2mm**:

- Elements of the previous two cases
- $(m\mathbf{a}_3 + n\mathbf{b}_3, I)(\mathbf{0}, -I) = (m\mathbf{a}_3 + n\mathbf{b}_3, -I)$
  Half-turns centred at the points $\frac{1}{4}n(2m+n)\mathbf{a}_3 + \frac{1}{4}(2\mathbf{b}_3 - \mathbf{a}_3)$

4.4 Case (iv) - The lattice of G is Square

We have $\mathbf{a}_4 = \begin{pmatrix} x \\ 0 \end{pmatrix}$ $\mathbf{b}_4 = \begin{pmatrix} 0 \\ x \end{pmatrix}$ where $x > 0$.

Theorem 4.8 If G is a wallpaper group corresponding to a pattern with a square lattice, then $J \subseteq \{I, A_{\frac{\pi}{2}}, -I, A_{\frac{3\pi}{2}}, B_0, B_{\frac{\pi}{2}}, B_\pi, B_{\frac{3\pi}{2}}\}$.

Proof. Possible rotations in J: I, $A_{\frac{\pi}{2}}$, $-I$, $A_{\frac{3\pi}{2}}$. Suppose $A_\theta \in J$. Then

$$A_\theta \mathbf{a} = (\cos\theta)\,\mathbf{a} + (\sin\theta)\,\mathbf{b} = k\mathbf{a} + l\mathbf{b}$$

And

$$A_\theta \mathbf{b} = (-\sin\theta)\,\mathbf{a} + (\cos\theta)\,\mathbf{b} = m\mathbf{a} + n\mathbf{b}$$

Thus we have equations

$$\sin\theta = k = n \ (21) \qquad \cos\theta = l = -m \ (22)$$

We have $\theta = 0$, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ only.

Possible reflections in J: B_0, $B_{\frac{\pi}{2}}$, B_π, $B_{\frac{3\pi}{2}}$. Suppose $B_\phi \in J$. Then

$$B_\phi \mathbf{a} = (\cos\phi)\,\mathbf{a} + (\sin\phi)\,\mathbf{b} = k\mathbf{a} + l\mathbf{b}$$

And

$$B_\phi \mathbf{b} = (\sin\phi)\,\mathbf{a} + (-\cos\phi)\,\mathbf{b} = m\mathbf{a} + n\mathbf{b}$$

Thus we have equations

$$\sin\theta = l = m \ (23) \qquad \cos\theta = k = -n \ (24)$$

Only $\phi = 0$, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ satisfy both equations.

So we have $J \subseteq \{I, A_{\frac{\pi}{2}}, -I, A_{\frac{3\pi}{2}}, B_0, B_{\frac{\pi}{2}}, B_\pi, B_{\frac{3\pi}{2}}\}$ □

Thus J is one of: $\{I\}$, $\{I, -I\}$, $\{I, B_0\}$, $\{I, B_\pi\}$, $\{I, -I, B_0, B_\pi\}$, $\{I, B_{\frac{\pi}{2}}\}$, $\{I, B_{\frac{3\pi}{2}}\}$ or $\{I, A_{\frac{\pi}{2}}, -I, A_{\frac{3\pi}{2}}\}$ [2].

4.4.1 $J = \{I, B_0\}$

Choosing $\theta = 0$, $\bar{\mathbf{a}} = \mathbf{a}_4$, $\bar{\mathbf{b}} = \mathbf{b}_4$, $\alpha = m$, $\beta = n$, $\lambda = 1$, $\mu = \frac{y}{x}$, $k = m$, $l = n$ in Theorem 4.4 gives an isomorphism from the resulting groups to those of case 4.2.1.

4.4.2 $J = \{I, B_\pi\}$

The isomorphism of case 4.4.1 maps the resulting groups to those of case 4.2.2.

4.4.3 $J = \{I, -I, B_0, B_\pi\}$

The isomorphism of case 4.4.1 maps the resulting groups to those of case 4.2.3.

4.4.4 $J = \{I, B_{\frac{\pi}{2}}\}$

Choosing $\theta = \frac{\pi}{4}$, $\bar{\mathbf{a}} = A_{-\theta}(\mathbf{a}_4 + \mathbf{b}_4)$, $\bar{\mathbf{b}} = A_{-\theta}(\mathbf{a}_4 - \mathbf{b}_4)$, $\alpha = \frac{1}{2}(m+n)$, $\beta = \frac{1}{2}(m-n)$, $\lambda = \frac{1}{\sqrt{2}}$, $\mu = -\frac{y}{\sqrt{2}x}$, $k = n$, $l = m - n$ in Theorem 4.4 gives an isomorphism from the resulting group to that of case 4.3.1, i.e., **cm**.

4.4.5 $J = \{I, B_{\frac{3\pi}{2}}\}$

The isomorphism of case 4.4.4 maps this group to that of case 4.3.2.

4.4.6 $J = \{I, -I, B_{\frac{\pi}{2}}, B_{\frac{3\pi}{2}}\}$

The isomorphism of case 4.4.4 maps this group to **c2mm**.

4.4.7 $J = \{I, A_{\frac{\pi}{2}}, -I, A_{\frac{3\pi}{2}}\}$

Choose the centre of a rotation of order 4 as the origin so that $(\mathbf{0}, M) \in G$ for all $M \in J$ and we have **p4**.

Elements of **p4**:

- Translations
- $(m\mathbf{a}_4 + n\mathbf{b}_4, I)(\mathbf{0}, -I) = (m\mathbf{a}_4 + n\mathbf{b}_4, -I)$
  Half-turns centred at the points $\frac{1}{2}m\mathbf{a}_4 + \frac{1}{2}n\mathbf{b}_4$
- $(m\mathbf{a}_4 + n\mathbf{b}_4, I)(\mathbf{0}, A_{\frac{\pi}{2}}) = (m\mathbf{a}_4 + n\mathbf{b}_4, A_{\frac{\pi}{2}})$
  Anticlockwise rotations by $\frac{\pi}{2}$ centred at the points $\frac{1}{2}(m-n)\mathbf{a}_4 + \frac{1}{2}(m+n)\mathbf{b}_4$
- $(m\mathbf{a}_4 + n\mathbf{b}_4, I)(\mathbf{0}, A_{\frac{3\pi}{2}}) = (m\mathbf{a}_4 + n\mathbf{b}_4, A_{\frac{3\pi}{2}})$
  Anticlockwise rotations by $\frac{3\pi}{2}$ centred at the points $\frac{1}{2}(m+n)\mathbf{a}_4 + \frac{1}{2}(n-m)\mathbf{b}_4$

4.4.8 $J = \{I, A_{\frac{\pi}{2}}, -I, A_{\frac{3\pi}{2}}, B_0, B_{\frac{\pi}{2}}, B_\pi, B_{\frac{3\pi}{2}}\}$

Choose the center of the rotation of order 4 as origin so that $(\mathbf{0}, A) \in G$ for all $A \in \{I, A_{\frac{\pi}{2}}, -I, A_{\frac{3\pi}{2}}\}$. Suppose $(\alpha\mathbf{a}_4 + \beta\mathbf{b}_4, B_0)$ realises B_0 from J in G. (Note that we cannot assume $\beta = 0$ as we have not chosen the origin to be on the glide line.) Then

$$(\alpha\mathbf{a}_4 + \beta\mathbf{b}_4, B_0)(\alpha\mathbf{a}_4 + \beta\mathbf{b}_4, B_0) = (2\alpha\mathbf{a}_4, I)$$

so that $\alpha = \frac{1}{2}k$ for some $k \in \mathbb{Z}$ and $(\frac{1}{2}k + \beta\mathbf{b}_4, B_0) \in G$. The two cases, where k is odd or even, give rise to two wallpaper groups.

If k is even, $-\frac{1}{2}k \in \mathbb{Z}$ so

$$(-\tfrac{1}{2}k\mathbf{a}_4, I)(\tfrac{1}{2}k\mathbf{a}_4 + \beta\mathbf{b}_4, B_0) = (\beta\mathbf{b}_4, B_0) \in G$$

Then

$$\big((\mathbf{0}, A_{\frac{\pi}{2}})(\beta\mathbf{b}_4, B_0)\big)^2 = (-\beta\mathbf{a}_4 - \beta\mathbf{b}_4, I)$$

so that $\beta \in \mathbb{Z}$ and $(-\beta\mathbf{b}_4, I) \in G$. Thus

$$(-\beta\mathbf{b}_4, I)(\beta\mathbf{b}_4, B_0) = (\mathbf{0}, B_0) \in G$$

so that, since $(\mathbf{0}, A_{\frac{\pi}{2}}) \in G$, $(0, M) \in G$ for all $M \in J$ and we have **p4mm**.
Elements of **p4mm**:

- Elements of **p4**
- $(m\mathbf{a}_4 + n\mathbf{b}_4, I)(\mathbf{0}, B_0) = (m\mathbf{a}_4 + 2(\frac{1}{2}n\mathbf{b}_4), B_0)$
  Horizontal trivial glides passing through and midway between the lattice points with translation part a multiple of $m\mathbf{a}_4$
- $(m\mathbf{a}_4 + n\mathbf{b}_4, I)(\mathbf{0}, B_{\frac{\pi}{2}}) =$
  $\big(\frac{1}{2}(m+n)(\mathbf{a}_4 + \mathbf{b}_4) + 2(\frac{1}{4}(m-n)(\mathbf{a}_4 - \mathbf{b}_4)), B_{\frac{\pi}{2}}\big)$
  Glides at $\frac{\pi}{4}$ to the horizontal which are trivial and pass through the lattice points with translation part a multiple of $(\mathbf{a}_4 - \mathbf{b}_4)$, or are nontrivial and pass midway between the lattice points with translation part an odd multiple of $\frac{1}{2}(\mathbf{a}_4 + \mathbf{b}_4)$
- $(m\mathbf{a}_4 + n\mathbf{b}_4, I)(\mathbf{0}, B_\pi) = \big(2(\frac{1}{2}m\mathbf{a}_4) + n\mathbf{b}_4), B_\pi\big)$
  Horizontal trivial glides passing through and midway between the lattice points with translation part a multiple of $m\mathbf{b}_4$
- $(m\mathbf{a}_4 + n\mathbf{b}_4, I)(\mathbf{0}, B_{\frac{3\pi}{2}}) =$
  $\Big(\frac{1}{2}(m-n)(\mathbf{a}_4 - \mathbf{b}_4) + 2(\frac{1}{4}(m+n)(\mathbf{a}_4 + \mathbf{b}_4)), B_{\frac{3\pi}{2}}\Big)$
  Glides at $\frac{3\pi}{4}$ to the horizontal which are trivial and pass through the lattice points with translation part a multiple of $(\mathbf{a}_4 - \mathbf{b}_4)$, or are nontrivial and pass midway between the lattice points with translation part an odd multiple of $\frac{1}{2}(\mathbf{a}_4 + \mathbf{b}_4)$

If k is odd, $\frac{1}{2}(1-k) \in \mathbb{Z}$ so

$$\left(\tfrac{1}{2}(1-k)\mathbf{a}_4, I\right)\left(\tfrac{1}{2}k\mathbf{a}_4 + \beta\mathbf{b}_4, B_0\right) = \left(\tfrac{1}{2}\mathbf{a}_4 + \beta\mathbf{b}_4, B_0\right) \in G$$

Then

$$\left((\mathbf{0}, A_{\frac{\pi}{2}})(\tfrac{1}{2}\mathbf{a}_4 + \beta\mathbf{b}_4, B_0)\right)^2 = \left((\tfrac{1}{2}-\beta)\mathbf{a}_4 + (\tfrac{1}{2}-\beta)\mathbf{b}_4, I\right)$$

so $\frac{1}{2} - \beta \in \mathbb{Z}$. Hence

$$((\tfrac{1}{2}-\beta)\mathbf{b}_4, I)(\tfrac{1}{2}\mathbf{a}_4 + \beta\mathbf{b}_4, B_0) = (\tfrac{1}{2}\mathbf{a}_4 + \tfrac{1}{2}\mathbf{b}_4, B_0) \in G$$

so that, since $(\mathbf{0}, A_{\frac{\pi}{2}}) \in G$, $\left(\frac{1}{2}\mathbf{a}_4 + \frac{1}{2}\mathbf{b}_4, M\right) \in G$ for all $M \in J$ and we have **p4gm**.

Elements of **p4gm**:

- Elements of **p4**
- $(m\mathbf{a}_4 + n\mathbf{b}_4, I)(\frac{1}{2}\mathbf{a}_4 + \frac{1}{2}\mathbf{b}_4, B_0) = \left(\frac{1}{2}(2m+1)\mathbf{a}_4 + 2(\frac{1}{4}(2n+1)\mathbf{b}_4), B_0\right)$
  Horizontal nontrivial glides passing through odd multiples of $\frac{1}{4}\mathbf{b}_4$ with translation part an odd multiple of $\frac{1}{2}\mathbf{a}_4$
- $(m\mathbf{a}_4 + n\mathbf{b}_4, I)(\frac{1}{2}\mathbf{a}_4 + \frac{1}{2}\mathbf{b}_4, B_\pi) = \left(\frac{1}{2}(2n+1)\mathbf{b}_4 + 2(\frac{1}{4}(2m+1)\mathbf{a}_4), B_\pi\right)$
  Vertical nontrivial glides passing through odd multiples of $\frac{1}{4}\mathbf{a}_4$ with translation part an odd multiple of $\frac{1}{2}\mathbf{b}_4$
- $(m\mathbf{a}_4 + n\mathbf{b}_4, I)\left(\frac{1}{2}\mathbf{a}_4 + \frac{1}{2}\mathbf{b}_4, B_{\frac{\pi}{2}}\right)$
  $= \left(\frac{1}{2}(m+n+1)(\mathbf{a}_4 + \mathbf{b}_4) + 2(\frac{1}{4}(m-n)(\mathbf{a}_4 - \mathbf{b}_4)), B_{\frac{\pi}{2}}\right)$
  Glides at $\frac{\pi}{4}$ to the horizontal which are nontrivial and pass through the lattice points with translation part an odd multiple of $\frac{1}{2}(\mathbf{a}_4 + \mathbf{b}_4)$, or are trivial and pass midway between the lattice points with translation part a multiple of $\mathbf{a}_4 + \mathbf{b}_4$
- $(m\mathbf{a}_4 + n\mathbf{b}_4, I)(\frac{1}{2}\mathbf{a}_4 + \frac{1}{2}\mathbf{b}_4, B_{\frac{3\pi}{2}})$
  $= \left(\frac{1}{2}(m-n)(\mathbf{a}_4 - \mathbf{b}_4) + 2(\frac{1}{4}(m+n+1)(\mathbf{a}_4 + \mathbf{b}_4)), B_{\frac{3\pi}{2}}\right)$
  Glides at $\frac{3\pi}{4}$ to the horizontal which are nontrivial and pass through the lattice points with translation part an odd multiple of $\frac{1}{2}(\mathbf{a}_4 - \mathbf{b}_4)$, or are trivial and pass midway between the lattice points with translation part a multiple of $\mathbf{a}_4 - \mathbf{b}_4$

4.5 Case (v) - The lattice of G is Hexagonal

We have $\mathbf{a}_5 = \begin{pmatrix} x \\ 0 \end{pmatrix}$ $\mathbf{b}_5 = \begin{pmatrix} \frac{1}{2}x \\ \frac{\sqrt{3}}{2}x \end{pmatrix}$ where $x > 0$.

Theorem 4.9 If G is a wallpaper group corresponding to a pattern with a square lattice, then $J \subseteq \{I,\ A_{\frac{\pi}{3}},\ A_{\frac{2\pi}{3}},\ -I,\ A_{\frac{4\pi}{3}},\ A_{\frac{5\pi}{3}},\ B_0,\ B_{\frac{\pi}{3}},\ B_{\frac{2\pi}{3}},\ B_\pi,\ B_{\frac{4\pi}{3}},\ B_{\frac{5\pi}{3}}\}$.

Proof. Possible rotations in J: I, $A_{\frac{\pi}{3}}$, $A_{\frac{2\pi}{3}}$, $-I$, $A_{\frac{4\pi}{3}}$, $A_{\frac{5\pi}{3}}$. Suppose $A_\theta \in J$. Then

$$A_\theta \mathbf{a}_5 = \left(\cos\theta - \tfrac{1}{\sqrt{3}}\sin\theta\right)\mathbf{a}_5 + \left(\tfrac{2}{\sqrt{3}}\sin\theta\right)\mathbf{b}_5 = k\mathbf{a}_5 + l\mathbf{b}_5$$

And

$$A_\theta \mathbf{b}_5 = \left(-\tfrac{2}{\sqrt{3}}\sin\theta\right)\mathbf{a}_5 + \left(\tfrac{1}{\sqrt{3}}\sin\theta + \cos\theta\right)\mathbf{b}_5 = m\mathbf{a}_5 + n\mathbf{b}_5$$

Thus we have equations

$$\begin{array}{cc} \cos\theta - \tfrac{1}{\sqrt{3}}\sin\theta = k \ (25) & \tfrac{2}{\sqrt{3}}\sin\theta = l \ (26) \\ -\tfrac{2}{\sqrt{3}}\sin\theta = m \ (27) & \tfrac{1}{\sqrt{3}}\sin\theta + \cos\theta = n \ (28) \end{array}$$

Since $-2 < -\frac{2}{\sqrt{3}} < \frac{2}{\sqrt{3}}\sin\theta < \frac{2}{\sqrt{3}}\sin\theta < 2$, (23) implies $\frac{2}{\sqrt{3}}\sin\theta = 0, \pm 1 \Rightarrow \sin\theta = 0, \pm\frac{\sqrt{3}}{2} \Rightarrow \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$ which satisfy all four equations.

Possible reflections in J: B_0, $B_{\frac{\pi}{3}}$, $B_{\frac{2\pi}{3}}$, B_π, $B_{\frac{4\pi}{3}}$, $B_{\frac{5\pi}{3}}$. Suppose $B_\phi \in J$. Then

$$B_\phi \mathbf{a}_5 = \left(\cos\phi - \tfrac{1}{\sqrt{3}}\sin\phi\right)\mathbf{a}_5 + \left(\tfrac{2}{\sqrt{3}}\sin\phi\right)\mathbf{b}_5 = k\mathbf{a}_5 + l\mathbf{b}_5$$

And

$$B_\phi \mathbf{b}_5 = \left(\cos\phi + \tfrac{1}{\sqrt{3}}\sin\phi\right)\mathbf{a}_5 + \left(\tfrac{1}{\sqrt{3}}\sin\phi - \cos\phi\right)\mathbf{b}_5 = m\mathbf{a}_5 + n\mathbf{b}_5$$

Thus we have equations

$$\begin{array}{cc} \cos\phi - \tfrac{1}{\sqrt{3}}\sin\phi = k \ (29) & \tfrac{2}{\sqrt{3}}\sin\phi = l \ (30) \\ \cos\phi + \tfrac{1}{\sqrt{3}}\sin\phi = m \ (31) & \tfrac{1}{\sqrt{3}}\sin\phi - \cos\phi = n \ (32) \end{array}$$

Only $\phi = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$ satisfy all four equations.

So we have $J \subseteq \{I, A_{\frac{\pi}{3}}, A_{\frac{2\pi}{3}}, -I, A_{\frac{4\pi}{3}}, A_{\frac{5\pi}{3}}, B_0, B_{\frac{\pi}{3}}, B_{\frac{2\pi}{3}}, B_\pi, B_{\frac{4\pi}{3}}, B_{\frac{5\pi}{3}}\}$ □

Thus J is one of: $\{I\}$, $\{I, -I\}$, $\{I, B_0\}$, $\{I, B_{\frac{\pi}{3}}\}$, $\{I, B_{\frac{2\pi}{3}}\}$, $\{I, B_\pi\}$, $\{I, B_{\frac{4\pi}{3}}\}$, $\{I, B_{\frac{5\pi}{3}}\}$, $\{I, -I, B_0, B_\pi\}$, $\{I, -I, B_{\frac{\pi}{3}}, B_{\frac{4\pi}{3}}\}$, $\{I, B_{\frac{2\pi}{3}}, B_{\frac{5\pi}{3}}\}$, $\{I, A_{\frac{2\pi}{3}}, A_{\frac{4\pi}{3}}\}$, $\{I, A_{\frac{2\pi}{3}}, A_{\frac{4\pi}{3}}, B_0, B_{\frac{2\pi}{3}}, B_{\frac{4\pi}{3}}\}$, $\{I, A_{\frac{2\pi}{3}}, A_{\frac{4\pi}{3}}, B_{\frac{\pi}{3}}, B_\pi, B_{\frac{5\pi}{3}}\}$, $\{I, A_{\frac{\pi}{3}}, A_{\frac{2\pi}{3}}, -I, A_{\frac{4\pi}{3}}, A_{\frac{5\pi}{3}}\}$
or $\{I, A_{\frac{\pi}{3}}, A_{\frac{2\pi}{3}}, -I, A_{\frac{4\pi}{3}}, A_{\frac{5\pi}{3}}, B_0, B_{\frac{\pi}{3}}, B_{\frac{2\pi}{3}}, B_\pi, B_{\frac{4\pi}{3}}, B_{\frac{5\pi}{3}}\}$ [2].

4.5.1 $J = \{I, B_0\}$

Choosing $\theta = 0$, $\bar{\mathbf{a}} = \mathbf{a}_5$, $\bar{\mathbf{b}} = 2\mathbf{b}_5 - \mathbf{a}_5$, $\alpha = \frac{1}{2}(2m+n)$, $\beta = \frac{1}{2}n$, $\lambda = 1$, $\mu = d\frac{y}{\sqrt{3}x}$, $k = m$, $l = n$ in Theorem 4.4 gives an isomorphism from the resulting group to that of case 4.3.1.

4.5.2 $J = \{I, B_{\frac{\pi}{3}}\}$

Choosing $\theta = \frac{\pi}{6}$, $\bar{\mathbf{a}} = A_{-\frac{\pi}{6}}(\mathbf{a}_5 + \mathbf{b}_5)$, $\bar{\mathbf{b}} = A_{-\frac{\pi}{6}}(\mathbf{b}_5 - \mathbf{a}_5)$, $\alpha = \frac{1}{2}(m+n)$, $\beta = \frac{1}{2}n$, $\lambda = \frac{1}{\sqrt{3}}$, $\mu = \dfrac{y}{x}$, $k = m$, $l = n$ in Theorem 4.4 gives an isomorphism from the resulting group to that of case 4.3.1.

4.5.3 $J = \{I, B_{\frac{2\pi}{3}}\}$

Choosing $\theta = \frac{\pi}{3}$, $\bar{\mathbf{a}} = A_{-\frac{\pi}{3}}\mathbf{b}_5$, $\bar{\mathbf{b}} = A_{-\frac{\pi}{3}}(2\mathbf{a}_5 - \mathbf{b}_5)$, $\alpha = \frac{1}{2}(2n+m$, $\beta = \frac{1}{2}m$, $\lambda = 1$, $\mu = -\dfrac{y}{\sqrt{3}x}$, $k = n$, $l = m$ in Theorem 4.4 gives an isomorphism from the resulting group to that of case 4.3.1.

4.5.4 $J = \{I, B_{\pi}\}$

The isomorphism of Case 4.5.1 maps the resulting group to that of case 4.3.2.

4.5.5 $J = \{I, B_{\frac{4\pi}{3}}\}$

The isomorphism of Case 4.5.2 maps the resulting group to that of case 4.3.2.

4.5.6 $J = \{I, B_{\frac{5\pi}{3}}\}$

The isomorphism of Case 4.5.3 maps the resulting group to that of case 4.3.2.

4.5.7 $J = \{I, -I, B_0, B_{\pi}\}$

The isomorphism of Case 4.5.1 maps the resulting group to that of case 4.3.3.

4.5.8 $J = \{I, -I, B_{\frac{\pi}{3}}, B_{\frac{4\pi}{3}}\}$

The isomorphism of Case 4.5.2 maps the resulting group to that of case 4.3.3.

4.5.9 $J = \{I, -I, B_{\frac{2\pi}{3}}, B_{\frac{5\pi}{3}}\}$

The isomorphism of Case 4.5.3 maps the resulting group to that of case 4.3.3.

4.5.10 $J = \{I, A_{\frac{2\pi}{3}}, A_{\frac{4\pi}{3}}\}$

Choose the centre of rotation order 3 as origin so that $(\mathbf{0}, M) \in G$ for all $M \in J$.
Elements of **p3**:

- Translations
- $(m\mathbf{a}_5 + n\mathbf{b}_5, I)(\mathbf{0}, A_{\frac{2\pi}{3}}) = (m\mathbf{a}_5 + n\mathbf{b}_5, A_{\frac{2\pi}{3}})$
  Anticlockwise rotations by $\frac{2\pi}{3}$ centred at the points $\frac{1}{3}(m-n)\mathbf{a}_5 + \left(n + \frac{1}{3}(m-n)\right)\mathbf{b}_5$

- $(m\mathbf{a}_5 + n\mathbf{b}_5, I)(\mathbf{0}, A_{\frac{4\pi}{3}}) = (m\mathbf{a}_5 + n\mathbf{b}_5, A_{\frac{4\pi}{3}})$
  Anticlockwise rotations by $\frac{4\pi}{3}$ centred at the points $\left(m + \frac{1}{3}(n-m)\right)\mathbf{a}_5 + \frac{1}{3}(n-m)\mathbf{b}_5$

4.5.11 $J = \{I,\ A_{\frac{2\pi}{3}},\ A_{\frac{4\pi}{3}},\ B_0,\ B_{\frac{2\pi}{3}},\ B_{\frac{4\pi}{3}}\}$

Choose the centre of a rotation order 3 as the origin so that $(\mathbf{0}, A) \in G$ for all $A \in \{I,\ A_{\frac{2\pi}{3}},\ A_{\frac{4\pi}{3}}\}$. Suppose $(\alpha(\mathbf{a}_5 - \mathbf{b}_5) + \beta\mathbf{b}_5, B_0)$ realises B_0 from J in G. Then

$$(\alpha(\mathbf{a}_5 - \mathbf{b}_5) + \beta\mathbf{b}_5, B_0)^2 = ((\alpha+\beta)(\mathbf{a}_5 - \mathbf{b}_5) + (\alpha+\beta)\mathbf{b}_5, I)$$

so $\alpha + \beta = k \in \mathbb{Z}$.
Also,

$$\left((\mathbf{0}, A_{\frac{2\pi}{3}})(\alpha(\mathbf{a}_5 - \mathbf{b}_5) + \beta\mathbf{b}_5, B_0)\right)^2 = (\beta\mathbf{a}_5 + (\alpha - \beta)\mathbf{b}_5, I)$$

so that $\beta \in \mathbb{Z}$. Thus $\alpha = k - \beta \in \mathbb{Z}$ so that

$$(-\alpha\mathbf{a}_5 + (\alpha-\beta)\mathbf{b}_5, I)\,(\alpha(\mathbf{a}_5 - \mathbf{b}_5) + \beta\mathbf{b}_5, B_0) = (\mathbf{0}, B_0) \in G$$

and we conclude that $(\mathbf{0}, M) \in G$ for all $M \in J$ and we have **p3m1**.
Elements of **p3m1**:

- Elements of p3
- $(m\mathbf{a}_5 + n\mathbf{b}_5, B_0) = \left(\frac{1}{2}(2m+n)\mathbf{a}_5 + \frac{1}{2}n(2\mathbf{b}_5 - \mathbf{a}_5), B_0\right)$
  Horizontal glides which are trivial and pass through the lattice points with translation part a multiple of $\mathbf{a}_5$, or are nontrivial and pass midway between the lattice points with translation part an odd multiple of $\frac{1}{2}\mathbf{a}_5$
- $(m\mathbf{a}_5 + n\mathbf{b}_5, B_{\frac{2\pi}{3}}) = \left(\frac{1}{2}(2n+m)\mathbf{b}_5 + \frac{1}{2}m(2\mathbf{a}_5 - \mathbf{b}_5), B_{\frac{2\pi}{3}}\right)$
  Glides at $\frac{\pi}{3}$ to the horizontal which are trivial and pass through the lattice points with translation part a multiple of $\mathbf{b}_5$, or are nontrivial and pass midway between the lattice points with translation part an odd multiple of $\frac{1}{2}\mathbf{b}_5$
- $(m\mathbf{a}_5 + n\mathbf{b}_5, B_{\frac{4\pi}{3}}) = \left(\frac{1}{2}(n-m)(\mathbf{b}_5 - \mathbf{a}_5) + \frac{1}{2}(m+n)(\mathbf{a}_5 + \mathbf{b}_5), B_{\frac{4\pi}{3}}\right)$
  Glides at $\frac{2\pi}{3}$ to the horizontal which are trivial and pass through the lattice points with translation part a multiple of $\mathbf{b}_5 - \mathbf{a}_5$, or are nontrivial and pass midway between the lattice points with translation part an odd multiple of $\frac{1}{2}(\mathbf{b}_5 - \mathbf{a}_5)$

4.5.12 $J = \{I,\ A_{\frac{2\pi}{3}},\ A_{\frac{4\pi}{3}},\ B_{\frac{\pi}{3}},\ B_{\pi},\ B_{\frac{5\pi}{3}}\}$

Choose the centre of a rotation of order 3 as the origin so that $(\mathbf{0}, A) \in G$ for all $A \in \{I,\ A_{\frac{2\pi}{3}},\ A_{\frac{4\pi}{3}}\}$. Suppose $(\alpha\mathbf{a}_5 + \beta\mathbf{b}_5, B_{\frac{4\pi}{3}})$ realises $B_{\frac{\pi}{3}}$ in G. Then

$$\left(\alpha\mathbf{a}_5+\beta\mathbf{b}_5, B_{\frac{\pi}{3}}\right)^2 = ((\alpha+\beta)\mathbf{a}_5+(\alpha+\beta)\mathbf{b}_5, I) \in G$$

so that $\alpha+\beta = k$ for $k \in \mathbb{Z}$.

Also $\left((\mathbf{0}, A_{\frac{2\pi}{3}})(\alpha\mathbf{a}_5+\beta\mathbf{b}_5, B_{\frac{\pi}{3}})\right)^2 = (-\alpha\mathbf{a}_5+2\alpha\mathbf{b}_5, I)$ G so that $-\alpha \in \mathbb{Z}$.

Hence $\beta = k-\alpha \in \mathbb{Z}$ so that $(-\alpha\mathbf{a}_5-\beta\mathbf{b}_5, I) \in G$ and thus

$$(-\alpha\mathbf{a}_5-\beta\mathbf{b}_5, I)(\alpha\mathbf{a}_5+\beta\mathbf{b}_5, B_{\frac{\pi}{3}}) = (\mathbf{0}, B_{\frac{\pi}{3}}) \in G.$$

We conclude that $(\mathbf{0}, M) \in G$ for all $M \in J$ and we have **p31m**.

Elements of **p31m**:

- Elements of **p3**
- $(m\mathbf{a}_5+n\mathbf{b}_5, B_{\frac{\pi}{3}}) = \left(\frac{1}{2}(m+n)(\mathbf{a}_5+\mathbf{b}_5)+\frac{1}{2}(n-m)(\mathbf{b}_5-\mathbf{a}_5), B_{\frac{\pi}{3}}\right)$
  Glides at $\frac{\pi}{6}$ to the horizontal which are trivial and pass through the lattice points with translation part a multiple of $\mathbf{a}_5+\mathbf{b}_5$, or are nontrivial and pass midway between the lattice points with translation part an odd multiple of $\frac{1}{2}(\mathbf{a}_5+\mathbf{b}_5)$
- $(m\mathbf{a}_5+n\mathbf{b}_5, B_{\pi}) = \left(\frac{1}{2}n(2\mathbf{b}_5-\mathbf{a}_5)+\frac{1}{2}(2m+n)\mathbf{a}_5, B_{\pi}\right)$
  Glides at $\frac{\pi}{2}$ to the horizontal which are trivial and pass through the lattice points with translation part a multiple of $2\mathbf{b}_5-\mathbf{a}_5$, or are nontrivial and pass midway between the lattice points with translation part an odd multiple of $\frac{1}{2}(2\mathbf{b}_5-\mathbf{a}_5)$
- $(m\mathbf{a}_5+n\mathbf{b}_5, B_{5\frac{\pi}{3}}) = \left(\frac{1}{2}n(2\mathbf{a}_5-\mathbf{b}_5)+\frac{1}{2}(2m+n)\mathbf{b}_5, B_{\frac{5\pi}{3}}\right)$
  Glides at $\frac{5\pi}{6}$ to the horizontal which are trivial and pass through the lattice points with translation part a multiple of $2\mathbf{a}_5-\mathbf{b}_5$, or are nontrivial and pass midway between the lattice points with translation part an odd multiple of $\frac{1}{2}(2\mathbf{a}_5-\mathbf{b}_5)$

4.5.13 $J = \{I,\ A_{\frac{\pi}{3}},\ A_{\frac{2\pi}{3}},\ -I,\ A_{\frac{4\pi}{3}},\ A_{\frac{5\pi}{3}}\}$

Choose the centre of a rotation of order 6 as the origin so that $(\mathbf{0}, M) \in G$ for all $M \in J$, and we have **p6**.

Elements of **p6**:

- Elements of **p3**
- $(m\mathbf{a}_5+n\mathbf{b}_5, I)(\mathbf{0}, A_{\frac{\pi}{3}}) = (m\mathbf{a}_5+n\mathbf{b}_5, A_{\frac{\pi}{3}})$
  Anticlockwise rotations by $\frac{\pi}{3}$ centred at the points $(m+n)\mathbf{b}-n\mathbf{a}$
- $(m\mathbf{a}_5+n\mathbf{b}_5, I)(\mathbf{0}, -I) = (m\mathbf{a}_5+n\mathbf{b}_5, -I)$
  Half-turns centred at the points $\frac{1}{2}m\mathbf{a}_5+\frac{1}{2}n\mathbf{b}_5$
- $(m\mathbf{a}_5+n\mathbf{b}_5, I)(\mathbf{0}, A_{\frac{5\pi}{3}}) = (m\mathbf{a}_5+n\mathbf{b}_5, A_{\frac{5\pi}{3}})$
  Anticlockwise rotations by $\frac{5\pi}{3}$ centred at the points $(m+n)\mathbf{a}-m\mathbf{b}$

4.5.14 $J = \{I,\ A_{\frac{\pi}{3}},\ A_{\frac{2\pi}{3}},\ -I,\ A_{\frac{4\pi}{3}},\ A_{\frac{5\pi}{3}},\ B_0,\ B_{\frac{\pi}{3}},\ B_{\frac{2\pi}{3}},\ B_{\pi},\ B_{\frac{4\pi}{3}},\ B_{\frac{5\pi}{3}}\}$

By the argument given for **p3m1**, $(\mathbf{0}, B_0) \in G$. Hence $(\mathbf{0}, M) \in G$ for all $M \in J$ and we have **p6mm**.

Elements of **p6mm**:

- All the elements of **p6**, **p3m1** and **p31m**.

4.6 The derived wallpaper groups are mutually non-isomorphic

By the contrapositive of Corollary 2.11, wallpaper groups with non-isomorphic point groups are not isomorphic, so we are left to show that wallpaper groups with point groups of the same size are not isomorphic. We apply Theorem 2.10 to do so.

Theorem 4.10 No two of **p2**, **pm**, **pg**, **cm** are isomorphic.

Proof. Of these only **p2** contains rotations so is not isomorphic to the others. Then **pg** is not isomorphic to **pm** nor **cm** since only **pg** does not contain reflections. Now, **cm** contains nontrivial glides whereas **pm** does not, so these two are not isomorphic. □

Theorem 4.11 No two of **p2mm**, **p2mg**, **p2gg**, **c2mm**.

Proof. Only **p2gg** does not contain reflections so is not isomorphic to the others. Of the remaining groups only **p2mm** does not contain nontrivial glides. Finally, **c2mm** contains a half-turn whereas **p2mg** does not. □

Lemma 4.12 Let the wallpaper groups G and G_1 be isomorphic. Then a rotation which is the product of two reflections in G is mapped to a rotation which is the product of two reflections in G_1.

Proof. Let the rotation $(\mathbf{v}, A) \in G$ be the product of reflections $(\mathbf{x}_1, B_1)$ and $(\mathbf{x}_2, B_2)$ in G_1 and let $\varphi : G \to G_1$ be an isomorphism. Then

$$\varphi\left((\mathbf{v}, A)\right) = \varphi\left((\mathbf{x_1}, B_1)(\mathbf{x_2}, B_2)\right) = \varphi\left((\mathbf{x_1}, B_1)\right)\varphi\left((\mathbf{x_2}, B_2)\right)$$

By Theorem 2.10, $\varphi\left((\mathbf{v}, A)\right)$ is a rotation in G_1 and $\varphi\left((\mathbf{x_1}, B_1)\right)$ and $\varphi\left((\mathbf{x_2}, B_2)\right)$ are reflections in G_1. □

Theorem 4.13 **p4mm** is not isomorphic to **p4gm**.

Proof. Each rotation of order 4 in **p4mm** can be written as the product of two reflections in **p4mm**. For example,

$$(m\mathbf{a} + n\mathbf{b}, A_{\frac{\pi}{2}}) = (m\mathbf{a} + n\mathbf{b}, B_{\pi})(\mathbf{0}, B_{\frac{\pi}{2}})$$

However, the only elements in the point group of **p4gm** that can be realised as reflections are $B_{\frac{\pi}{2}}$ and $B_{\frac{3\pi}{2}}$. Since the product of these is either I or $-I$, the product of two reflections in **p4gm** cannot be a rotation of order 4. Thus, by Lemma 4.12, **p4mm** and **p4gm** are not isomorphic. □

Theorem 4.14 **p3m1** is not isomorphic to **p31m**.

Proof. Each rotation of order 3 in **p31m** can be written as the product of two reflections in **p3m1**. For example,

$$(m\mathbf{a} + n\mathbf{b}, A_{\frac{\pi}{3}}) = ((m + \tfrac{1}{2}n)\mathbf{a} + \tfrac{1}{2}n(2\mathbf{b} - \mathbf{a}), B_{\pi})(0, B_{\frac{5\pi}{3}}).$$

However, the only elements in the point group of **p4gm** that can be realised as reflections are B_0, $B_{\frac{2\pi}{3}}$ and $B_{\frac{4\pi}{3}}$ and the product of any two of these is a rotation of order 2. Thus, by Lemma 4.12, **p3m1** and **p31m** are not isomorphic. □

5 Conclusion

By branching from each lattice, to each point group, and finally to each element, we have considered every possible case and in doing so provided a complete proof of the existence of the 17 wallpaper groups. However, because our target audience has been post 200-level undergraduate students, we have not achieved the generality that is associated with a more formal approach such as that of [4]. Although this means the natural extension to 3-space, and the demonstration of the 320 "crystallographic groups", is less straightforward, it none-the-less shows the possibility of such a demonstration.

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Appendix

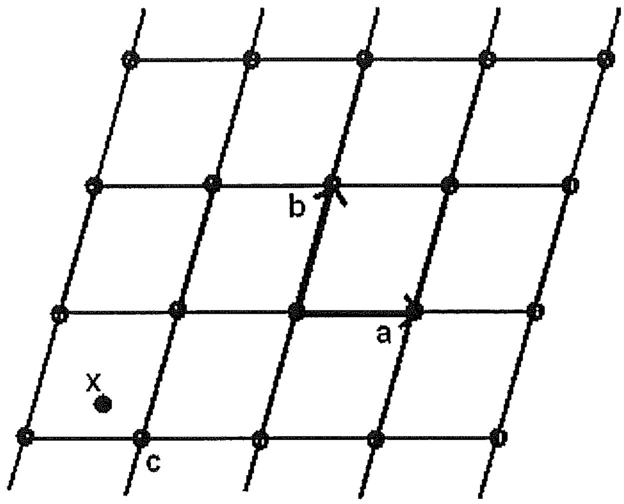

Figure 1: The points on the lattice divide the plane into parallelograms.

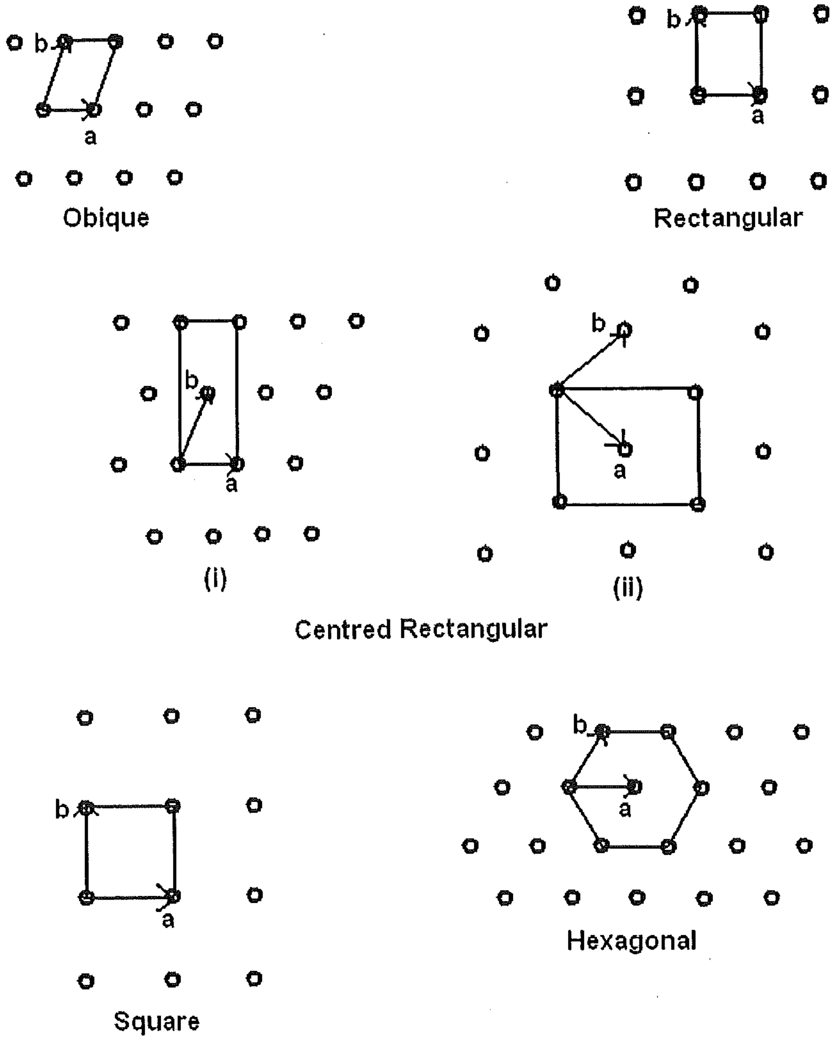

Figure 2: The fives types of lattices.

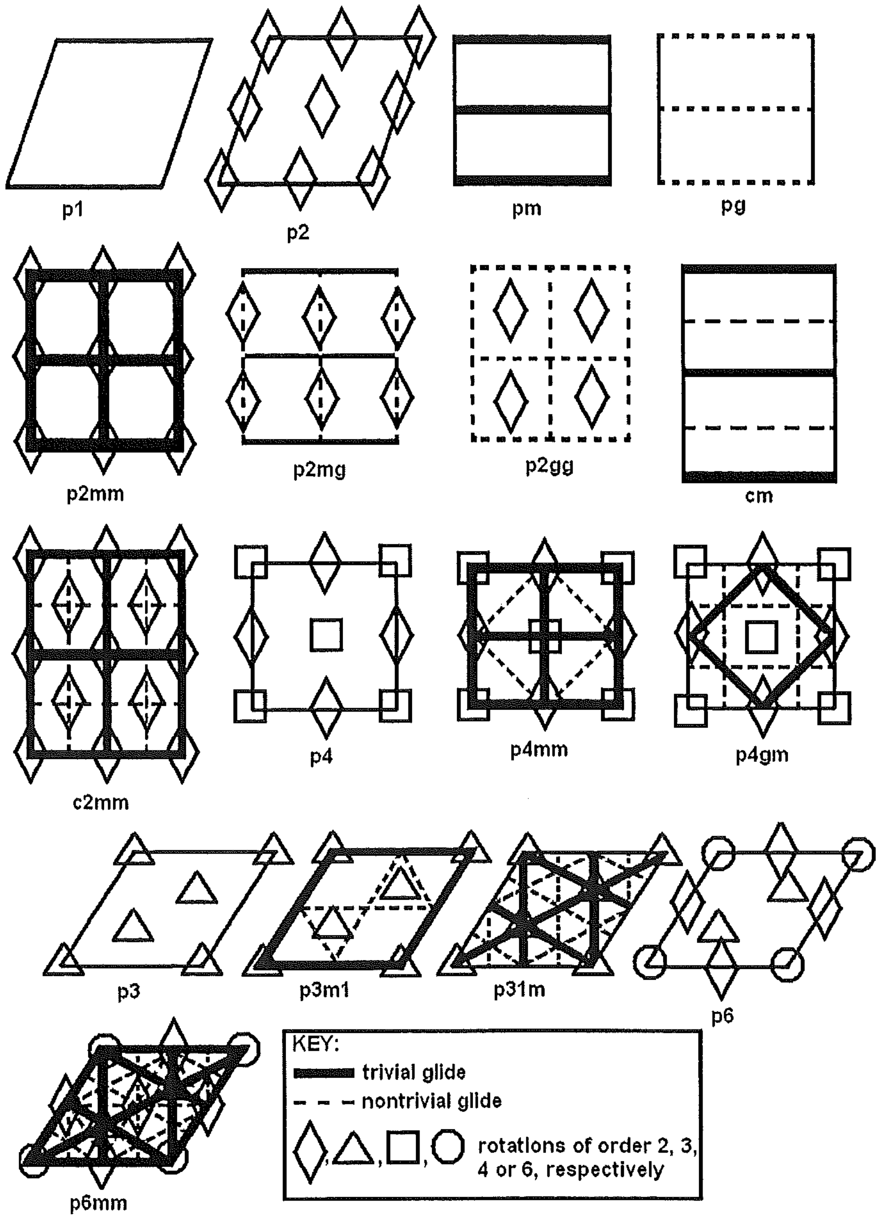

Figure 3: The seventeen wallpaper groups.